Or $\sum_{k=1}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt=\int_{\frac{1}{n}}^{\frac{2}{n}} f(t)dt+\int_{\frac{2}{n}}^{\frac{3}{n}} f(t)dt+\ldots+\int_{\frac{n-1}{n}}^{\frac{n}{n}} f(t)dt$

$= \int_{\frac{1}{n}}^{\frac{2}{n}} f(t)dt + \int_{\frac{2}{n}}^{\frac{3}{n}} f(t)dt +\ldots+\int_{\frac{n-1}{n}}^{1} f(t)dt$

$= \int_{\frac{1}{n}}^{1} f(t)dt = A\left(\frac{1}{n}\right)$ (relation de Chasles sur les intégrales)

Ainsi $\sum_{k=1}^{n-1}\frac{1}{n}f(\frac{k+1}{n}) \le A\left(\frac{1}{n}\right) \le \sum_{k=1}^{n-1}\frac{1}{n}f(\frac{k}{n})$.

2) On pose pour tout entier $n \ge 2, S_n = \frac{1}{n}\sum_{k=1}^{n-1} f(\frac{k}{n})$.

a- $\sum_{k=1}^{n-1}\frac{1}{n}f(\frac{k+1}{n}) \le A\left(\frac{1}{n}\right) \le \sum_{k=1}^{n-1}\frac{1}{n}f(\frac{k}{n})$.

$\Leftrightarrow \sum_{k=1}^{n-1}\frac{1}{n}f(\frac{k+1}{n}) \le A\left(\frac{1}{n}\right) \le S_n$

$* \sum_{k=1}^{n-1}\frac{1}{n}f(\frac{k+1}{n})=\frac{1}{n}\left[f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)+f\left(\frac{n}{n}\right)\right]$

$= \frac{1}{n}\left[f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)\right]$ car $f\left(\frac{n}{n}\right)=0$

$= \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\ldots+f\left(\frac{n-1}{n}\right)\right]-\frac{1}{n}f\left(\frac{1}{n}\right)$

$= S_n - \frac{1}{n}f\left(\frac{1}{n}\right)$.

Ainsi $S_n - \frac{1}{n}f\left(\frac{1}{n}\right) \le A\left(\frac{1}{n}\right) \Leftrightarrow S_n \le A\left(\frac{1}{n}\right)+\frac{1}{n}f\left(\frac{1}{n}\right)$

Conclusion: $A\left(\frac{1}{n}\right) \le S_n \le A\left(\frac{1}{n}\right)+\frac{1}{n}f\left(\frac{1}{n}\right)$.

b- • $\lim_{n\to+\infty} A\left(\frac{1}{n}\right)=\lim_{\lambda\to0^+} A(\lambda)=\frac{4}{3}$ (on pose $\lambda=\frac{1}{n}; n\to+\infty; \lambda=\frac{1}{n}\to 0^+$)

$* \lim_{n\to+\infty}\left[A\left(\frac{1}{n}\right)+\frac{1}{n}f\left(\frac{1}{n}\right)\right]=\lim_{\lambda\to0^+}[A(\lambda)+\lambda f(\lambda)]$ (on pose $\lambda=\frac{1}{n}$)

$=\lim_{\lambda\to0^+} A(\lambda)+\lim_{\lambda\to0^+}[\lambda^3-2\lambda\text{Log}\lambda-\lambda]$

$= \frac{4}{3}+0$ car $\lim_{\lambda\to0^+}(\lambda\text{Log}\lambda)=0$

$$\left.\begin{array}{l} A(\frac{1}{n}) \le S_n \le A\left(\frac{1}{n}\right)+\frac{1}{n}f\left(\frac{1}{n}\right); \forall n \ge 2 \\ \lim_{n\to+\infty} A\left(\frac{1}{n}\right)=\frac{4}{3} \\ \lim_{n\to+\infty}\left[A\left(\frac{1}{n}\right)+\frac{1}{n}f\left(\frac{1}{n}\right)\right]=\frac{4}{3} \end{array}\right\} \Rightarrow \lim_{n\to+\infty} S_n=\frac{4}{3}$$

3)a- • Vérifions la proposition pour n = 2